

# NON-INJECTIVITY OF SEIDEL'S MORPHISM

SÍLVIA ANJOS AND RÉMI LECLERCQ

**ABSTRACT.** The main purpose of this note is to exhibit a Hamiltonian diffeomorphism loop undetected by the Seidel morphism of certain 2-point blow-ups of  $S^2 \times S^2$ , only one of which being monotone. As side remarks, we show that Seidel's morphism is injective on all Hirzebruch surfaces, and discuss how to adapt the monotone example to the Lagrangian setting.

## 1. INTRODUCTION

The motivation for this work is the search of homotopy classes of loops of Hamiltonian diffeomorphisms which are not detected by Seidel's morphism. Given a symplectic manifold  $(M, \omega)$ , and its Hamiltonian diffeomorphism group  $\text{Ham}(M, \omega)$ , recall that Seidel's morphism

$$\mathcal{S}: \pi_1(\text{Ham}(M, \omega)) \rightarrow QH_*(M, \Pi)^\times$$

was defined on a covering of  $\pi_1(\text{Ham}(M, \omega))$  by Seidel in [25] for strongly semi-positive symplectic manifolds and then on the fundamental group itself and for any closed symplectic manifold by Lalonde–McDuff–Polterovich in [17].

The target space,  $QH_*(M, \Pi)^\times$ , is the group of invertible elements of the quantum homology of  $(M, \omega)$ . More precisely, the small quantum homology of  $(M, \omega)$  is  $QH_*(M, \Pi) = H_*(M) \otimes \Pi$  where  $\Pi = \Pi^{\text{univ}}[q, q^{-1}]$  with  $q$  a degree 2 variable and the ring  $\Pi^{\text{univ}}$  consisting of generalized Laurent series in a degree 0 variable  $t$ :

$$(1) \quad \Pi^{\text{univ}} := \left\{ \sum_{\kappa \in \mathbb{R}} r_\kappa t^\kappa \left| r_\kappa \in \mathbb{Q}, \text{ and } \forall c \in \mathbb{R}, \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty \right. \right\}.$$

Since its construction, Seidel's morphism has been successfully used to detect many Hamiltonian loops (see e.g [21] and references therein), and was extended or generalized to various situations (see e.g [15], [24], [13], [14], [11]). A particular extension consists of secondary-type invariants, whose construction is based on Seidel's construction after enriching Floer homology by considering Leray–Serre spectral sequences introduced by Barraud–Cornea [6], and which should detect loops undetected by Seidel's morphism [7]. However, there were no Hamiltonian loops with

---

*Date:* February 19, 2016.

*2010 Mathematics Subject Classification.* Primary 53D45; Secondary 57S05, 53D05.

*Key words and phrases.* symplectic geometry, Seidel morphism, toric symplectic manifolds, Hirzebruch surfaces.

The authors would like to thank Dusa McDuff for her interest and useful discussions. The first author is partially funded by FCT/Portugal through project PEst-OE/EEI/LA0009/2013 and project EXCL/MAT-GEO/0222/2012. The second author is partially supported by ANR Grant ANR-13-JS01-0008-01.

non-trivial homotopy class known to be undetected by Seidel's morphism (as far as we know). This short note intends to provide the first example of such a loop on a family of symplectic manifolds. Moreover, the example is explicit and thus can easily be used to test other constructions. Notice finally that this example can also be used to construct other examples (e.g by products, see [19]).

**First try: Symplectically aspherical manifolds.** Looking for elements in the kernel of the Seidel morphism, one might first consider symplectically aspherical manifolds, by which we mean that both the symplectic form and the first Chern class vanish on the second homotopy group of the manifold. Indeed, such manifolds have trivial Seidel morphism.

The geometric reason for this is that, by construction, the Seidel morphism of  $(M, \omega)$  counts pseudo-holomorphic section classes of a fibration over  $S^2$  with fiber  $(M, \omega)$ . The difference between two such classes is thus given by elements of  $\pi_2(M)$  admitting a pseudo-holomorphic representative, whose existence is prevented by symplectic asphericity.

Alternatively, this can be proved via purely algebraic methods, using the equivalent description of Seidel's morphism, as a representation of  $\pi_1(\text{Ham}(M, \omega))$  into the Floer homology of  $(M, \omega)$ . Given a loop of Hamiltonian diffeomorphisms, one gets an automorphism of  $HF_*(M, \omega)$  which can be shown to act trivially by playing around with the following facts:

- (i) Morse homology (the quantum homology of symplectically aspherical manifolds) is a ring over which Floer homology is a module.
- (ii) All involved morphisms (PSS, Seidel, continuation) are module morphisms.
- (iii) Any automorphism of Morse homology preserves the fundamental class, since it generates the top degree homology group.
- (iv) The fundamental class is the unit of the Morse homology ring.

This line of ideas, which goes back to Seidel, has been used by McDuff–Salamon in [22] to simplify Schwarz's original proof of invariance of spectral invariants. It has then been adapted by Leclercq in [18] to Lagrangian spectral invariants and to prove the triviality of the relative (i.e Lagrangian) Seidel morphism by Hu–Lalonde–Leclercq in [14] (see Lemma 5.5).

Now, even though aspherical manifolds seem to be ideal candidate, there are no homotopically non-trivial loops of Hamiltonian diffeomorphisms known to the authors in such manifolds...

**Second try: Symplectic toric manifolds.** Symplectic toric geometry provides a large class of natural examples of symplectic manifolds which are complicated enough to be interesting while simple enough so that many rather involved constructions can be explicitly performed. In [4], we computed the Seidel morphism on NEF toric 4-manifolds following work of McDuff and Tolman [23]. Recall that by definition  $(M, J)$  is a NEF pair if there are no  $J$ -pseudo-holomorphic spheres in  $M$  with negative first Chern number. This gave, *in the particular case of 4-dimensional toric manifolds*, an elementary and somehow purely symplectic way to

perform these computations previously obtained by Chan, Lau, Leung, and Tseng [8] (and using works by Fukaya, Oh, Ohta, and Ono [10], and González and Iritani [12]). We also showed that one could then deduce the Seidel morphism of some non-NEF symplectic manifolds and, as an example, we explicitated the computation for some Hirzebruch surface.

The easiest symplectic toric 4-manifolds for which we can explicit a non-trivial element in the kernel of the Seidel morphism are 2-point blow-ups of  $S^2 \times S^2$ . More precisely, start with the monotone product  $(S^2 \times S^2, \omega_1)^1$  on which we perform two blow-ups. Notice that the resulting symplectic manifold is monotone only when the respective sizes of the blow-ups coincide *and are equal to  $\frac{1}{2}$* .

In Section 3, we exhibit a specific loop of Hamiltonian diffeomorphisms whose homotopy class is in the kernel of Seidel's morphism if and only if the size of the two blow-ups coincide. Since this loop, obtained from two circle actions, can easily be seen to be non-trivial, this obviously yields a family of symplectic manifolds, only one of which being monotone, with non-injective Seidel morphism, i.e

**Theorem 1.1.** *The Seidel morphism of the 2-point blow-up of  $(S^2 \times S^2, \omega_1)$  with equal sizes of the blow-ups is not injective.*

In our search of undetected Hamiltonian loops, we realized that

**Theorem 1.2.** *Seidel's morphism is injective on all Hirzebruch surfaces.*

While this is not hard to prove and might be well-known to experts, we did not find it in the literature and thus include a proof in Section 2.

**Discussion on the adaptation to the Lagrangian setting.** As mentioned above, there is a relative (i.e Lagrangian) version of the Seidel morphism defined by Hu–Lalonde in [13] and further studied by Hu–Lalonde–Leclercq in [14]. There are two ways to adapt the example of Theorem 1.1 to the Lagrangian setting which we discuss here. (However, in order to keep this note short – and without too many technical details on the standard tools involved here –, we will not investigate these ideas further on here.)

First, let us remark that to get the Lagrangian version of the Seidel morphism, we need to consider a monotone Lagrangian of minimal Maslov at least 2. So, in what follows, we have in mind the only monotone symplectic manifold of the family mentioned above, i.e the monotone product  $S^2 \times S^2$  with area of each factor equals to 1 on which we perform two blow-ups of size  $\frac{1}{2}$ .

- *The first way to relate absolute and relative settings* is to consider the diagonal of the symplectic product. More precisely, let  $(M, \omega)$  be a monotone symplectic manifold. The diagonal  $\Delta \simeq M$  is a monotone Lagrangian of the product  $(M \times M, \omega \oplus (-\omega))$ , which we denote  $(\widehat{M}, \widehat{\omega})$  for short, with minimal Maslov number equal to twice the minimal first Chern number of  $(M, \omega)$  and thus greater than or equal to 2. This allows us to consider the Lagrangian Seidel morphism:

$$\mathcal{S}_\Delta : \pi_1(\text{Ham}(\widehat{M}, \widehat{\omega}), \text{Ham}_\Delta(\widehat{M}, \widehat{\omega})) \rightarrow QH_*(\Delta)^\times$$

---

<sup>1</sup>traditionally,  $\omega_\mu$  denotes the product symplectic form with total area  $\mu \geq 1$  on the first factor and area 1 on the second one

where  $\text{Ham}_\Delta$  denotes the subgroup of  $\text{Ham}$  formed by Hamiltonian diffeomorphisms which preserve  $\Delta$  and  $QH_*(\Delta)$  denotes the Lagrangian quantum homology of  $\Delta$ .

An element  $\phi \in \pi_1(\text{Ham}(M, \omega))$  generated by the Hamiltonian  $H: M \times [0, 1] \rightarrow \mathbb{R}$ , induces  $\widehat{\phi} \in \pi_1(\text{Ham}(\widehat{M}, \widehat{\omega}), \text{Ham}_\Delta(\widehat{M}, \widehat{\omega}))$ , generated by  $\widehat{F} = F \oplus 0: \widehat{M} \times [0, 1] \rightarrow \mathbb{R}$ . To get an element in the kernel of the Lagrangian Seidel morphism, it only remains to prove that:

- (i)  $\mathcal{S}(\phi) = \mathcal{S}_\Delta(\widehat{\phi})$  in  $QH_*(M, \omega) \simeq QH_*(\Delta)$ , and (ii)  $\widehat{\phi}$  is non zero.

Note that in (i), not only the quantum homologies are canonically identified but the chain complexes themselves coincide and this identification agrees with the PSS morphisms in the following sense:

$$\begin{array}{ccc} QH_*(M, \omega) & \xlongequal{\quad} & QH_*(\Delta) \\ \text{PSS} \downarrow & & \downarrow \text{PSS} \\ HF_*(H, J) & \xlongequal{\quad} & HF_*(\widehat{H}, \widehat{J}: \Delta) \end{array}$$

as proved in the monotone setting by Leclercq–Zapolsky in [20] ( $J$  denotes an almost complex structure on  $M$ , compatible with and tamed by  $\omega$ , while  $\widehat{J}$  denotes an almost complex structure on  $\widehat{M}$  adapted to  $J$ ). This makes us believe that (i) can be straightforwardly shown to hold.

On the other hand, proving (ii) will require some other technique.

- *The second way to the Lagrangian setting* is to use Albers’s comparison map between Hamiltonian and Lagrangian Floer homologies from [3], denoted below by  $\mathcal{A}$ , which relates the absolute and relative Seidel morphisms via the following commutative diagram (see [13]):

$$\begin{array}{ccccc} \pi_1(\text{Ham}(M, \omega)) & \longrightarrow & \pi_1(\text{Ham}(M, \omega), \text{Ham}_L(M, \omega)) & \longrightarrow & \pi_0(\text{Ham}_L(M, \omega)) \\ \mathcal{S} \downarrow & & \downarrow \mathcal{S}_L & & \\ HF_*(M, \omega) & \xrightarrow{\quad \mathcal{A} \quad} & HF_*(M, \omega; L) & & \end{array}$$

where  $L$  is a closed monotone Lagrangian of  $(M, \omega)$  with minimal Maslov number at least 2.

To get an interesting example via this method, one has to choose  $L$  such that  $HF_*(M, \omega; L) \neq 0$  and to prove (again) that the image of  $\phi \in \pi_1(\text{Ham}(M, \omega))$  in  $\pi_1(\text{Ham}(M, \omega), \text{Ham}_L(M, \omega))$  is non-trivial.

## 2. HIRZEBRUCH SURFACES

We will have to proceed in two steps as the “even” and “odd” Hirzebruch surfaces have to be dealt with separately. Note that the following descriptions coincide with the one adopted in [4] to which we refer for further details.

**2.1. Even Hirzebruch surfaces.** As is [4] we use the conventions and the description adopted in [5] for these surfaces. We recall that the toric “even” Hirzebruch surfaces  $(\mathbb{F}_{2k}, \omega_\mu)$ ,  $0 \leq k \leq \ell$  with  $\ell \in \mathbb{N}$  and  $\ell < \mu \leq \ell + 1$ , can be identified with the symplectic manifolds  $M_\mu = (S^2 \times S^2, \omega_\mu)$  where  $\omega_\mu$  is the split symplectic form with area  $\mu \geq 1$  for the first  $S^2$ -factor, and with area 1 for the second factor. The moment polytope of  $\mathbb{F}_{2k}$  is

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, x_2 + kx_1 \geq 0, x_2 - kx_1 \leq \mu - k\}.$$

Let  $\Lambda_{e_1}^{2k}$  and  $\Lambda_{e_2}^{2k}$  represent the circle actions whose moment maps are, respectively, the first and second components of the moment map associated to the torus action  $T_{2k}$  acting on  $\mathbb{F}_{2k}$ . We will also denote by  $\Lambda_{e_1}^{2k}$  and  $\Lambda_{e_2}^{2k}$  the corresponding generators in  $\pi_1(T_{2k})$ .

It is well known (see e.g [1]) that for  $k = 0$ ,  $\pi_1(\text{Ham}(\mathbb{F}_0, \omega_\mu)) = \mathbb{Z}/2\mathbb{Z}\langle \Lambda_{e_1}^0, \Lambda_{e_2}^0 \rangle$  and that for  $k \geq 1$ ,  $\pi_1(\text{Ham}(\mathbb{F}_{2k}, \omega_\mu)) = \mathbb{Z}/2\mathbb{Z}\langle \Lambda_{e_1}^0, \Lambda_{e_2}^0 \rangle \oplus \mathbb{Z}\langle \Lambda_{e_1}^{2k} \rangle$ .

Let  $B = [S^2 \times \{p\}]$  and  $F = [\{p\} \times S^2] \in H_2(S^2 \times S^2; \mathbb{Z})$  and denote  $u = B \otimes q$  and  $v = F \otimes q$  where  $q$  is the degree 2 variable entering into play in the definition of  $\Pi = \Pi^{\text{univ}}[q, q^{-1}]$  and  $\Pi^{\text{univ}}$  the ring of generalised Laurent series defined by (1). Then the (small) quantum homology group of degree 4, is given as the algebra

$$QH_4(\mathbb{F}_{2k}, \omega_\mu) = \Pi^{\text{univ}}[u, v] / \langle u^2 = t^{-1}, v^2 = t^{-\mu} \rangle.$$

Now, recall from [4, Section 5.3] that

$$\begin{aligned} \mathcal{S}(\Lambda_{e_1}^0) &= B \otimes qt^{\frac{1}{2}} = ut^{\frac{1}{2}}, \quad \mathcal{S}(\Lambda_{e_2}^0) = F \otimes qt^{\frac{\mu}{2}} = vt^{\frac{\mu}{2}}, \quad \text{and} \\ \mathcal{S}(\Lambda_{e_1}^{2k}) &= (B + F) \otimes qt^{\frac{1}{2}-\epsilon} = (u + v)t^{\frac{1}{2}-\epsilon} \quad \text{with } \epsilon = \frac{1}{6\mu}. \end{aligned}$$

Moreover, we have  $\mathcal{S}(\Lambda_{e_1}^0)^2 = \mathcal{S}(\Lambda_{e_2}^0)^2 = \mathbb{1}$  and

$$(2) \quad \mathcal{S}(\Lambda_{e_1}^{2k})^{-1} = (B - F) \otimes q \frac{t^{\frac{1}{2}+\epsilon}}{1 - t^{1-\mu}} = (u - v) \frac{t^{\frac{1}{2}+\epsilon}}{1 - t^{1-\mu}}.$$

*Proof of Theorem 1.2 for even Hirzebruch surfaces.* Since  $\Lambda_{e_1}^0$  and  $\Lambda_{e_2}^0$  are of order 2, any element in  $\pi_1(\text{Ham}(\mathbb{F}_{2k}, \omega_\mu))$  is of the form  $\varepsilon_1 \Lambda_{e_1}^0 + \varepsilon_2 \Lambda_{e_2}^0 + \ell \Lambda_{e_1}^{2k}$ , with  $\varepsilon_1$  and  $\varepsilon_2$  in  $\{0, 1\}$  and  $\ell \in \mathbb{Z}$ . Moreover, it is in the kernel of  $\mathcal{S}$  if and only if  $\mathcal{S}(\Lambda_{e_1}^{2k})^\ell = \mathcal{S}(\Lambda_{e_1}^0)^{\varepsilon_1} \mathcal{S}(\Lambda_{e_2}^0)^{\varepsilon_2}$  i.e  $\mathcal{S}(\Lambda_{e_1}^{2k})^\ell$  is either  $u$ ,  $v$ , or  $uv$ , up to a power of  $t$ .

Now for  $\ell \in \mathbb{N} \setminus \{0\}$ , expanding  $\mathcal{S}(\Lambda_{e_1}^{2k})^\ell$  thanks to the Binomial Theorem shows that necessarily it is of the form  $C_1 \cdot u + C_2 \cdot v$  or  $C_1 + C_2 \cdot uv$  where  $C_1$  and  $C_2$  are linear combinations of powers of  $t$  with positive rational coefficients (hence non zero). Thus  $\varepsilon_1 \Lambda_{e_1}^0 + \varepsilon_2 \Lambda_{e_2}^0 + \ell \Lambda_{e_1}^{2k} \notin \ker(\mathcal{S})$  for any  $\varepsilon_1$  and  $\varepsilon_2$  in  $\{0, 1\}$  and  $\ell \in \mathbb{N}$ .

For  $\ell < 0$ ,  $\mathcal{S}(\Lambda_{e_1}^{2k})^\ell$  is, by the Binomial Theorem together with (2), of the form

$$\frac{C'_1 \cdot u - C'_2 \cdot v}{(1 - t^{1-\mu})^{-\ell}} \quad \text{or} \quad \frac{C'_1 - C'_2 \cdot uv}{(1 - t^{1-\mu})^{-\ell}}$$

which as above shows that the same holds in this case. This implies that the only elements of  $\pi_1(\text{Ham}(\mathbb{F}_{2k}, \omega_\mu))$  which could be in  $\ker(\mathcal{S})$  are of the form  $\varepsilon_1 \Lambda_{e_1}^0 + \varepsilon_2 \Lambda_{e_2}^0$  so that  $\ker(\mathcal{S}) = \{0\}$ .  $\square$

**2.2. Odd Hirzebruch surfaces.** Similarly, “odd” Hirzebruch surfaces  $(\mathbb{F}_{2k-1}, \omega'_\mu)$ ,  $1 \leq k \leq \ell$  with  $\ell \in \mathbb{N}$  and  $\ell < \mu \leq \ell + 1$ , can be identified with the symplectic manifolds  $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \omega'_\mu)$  where the symplectic area of the exceptional divisor is  $\mu > 0$  and the area of the projective line is  $\mu + 1$ . Its moment polytope is

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \begin{array}{l} 0 \leq x_1 + x_2 \leq 1, \ x_2(k-1) + kx_1 \geq 0, \\ kx_2 + (k-1)x_1 \geq k - \mu - 1 \end{array} \right\}.$$

Let  $\Lambda_{e_1}^{2k-1}$  and  $\Lambda_{e_2}^{2k-1}$  represent the circle actions whose moment maps are, respectively, the first and the second component of the moment map associated to the torus action  $T_{2k-1}$  acting on  $\mathbb{F}_{2k-1}$ . As before, we will also denote by  $\Lambda_{e_1}^{2k-1}, \Lambda_{e_2}^{2k-1}$  the generators in  $\pi_1(T_{2k-1})$ .

$\mathbb{F}_1$  is Fano and we can show that  $\Lambda_{e_1}^{2k-1} = \Lambda_{e_2}^{2k-1} = (2k-1)\Lambda_{e_1}^1$ , using Karshon’s classification of Hamiltonian circle actions [16]. It follows from [1] and [2] that  $\pi_1(\text{Ham}(\mathbb{F}_{2k-1}, \omega'_\mu)) = \mathbb{Z}\langle \Lambda_{e_1}^1 \rangle$ .

Let  $B \in H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}; \mathbb{Z})$  denote the homology class of the exceptional divisor with self intersection  $-1$  and  $F$  the class of the fiber of the fibration  $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \rightarrow S^2$ . Let  $u_1 = (B + F) \otimes q$ ,  $u_2 = u_4 = F \otimes q$ , and  $u_3 = B \otimes q$ . The additive relations clearly are given by  $u_2 = u_4$  and  $u_1 = u_2 + u_3$  while the primitive relations yield  $u_1 u_3 = t^{-1}$  and  $u_2 u_4 = u_3 t^{-\mu}$ . If we set  $u = F \otimes q$  then it follows from [23, Theorem 5.2] that the quantum homology algebra is given by

$$QH_4(\mathbb{F}_{2k+1}, \omega'_\mu) = \Pi^{\text{univ}}[u] / (u^4 t^{2\mu} + u^3 t^\mu - t^{-1}).$$

Now, the normal vectors to the moment polytope of  $\mathbb{F}_1$  are given by  $\eta_1 = (1, 1)$ ,  $\eta_2 = (0, -1)$ ,  $\eta_3 = (-1, -1)$  and  $\eta_4 = (-1, 0)$  and it follows from [23, Theorem 1.10] that the Seidel elements of the actions  $\Gamma_i$  associated to  $\eta_i$  are given by

$$\begin{aligned} \mathcal{S}(\Gamma_1) &= (B + F) \otimes q t^{1+\mu-2\varepsilon} = u_1 t^{1+\mu-2\varepsilon}, \quad \mathcal{S}(\Gamma_2) = \mathcal{S}(\Gamma_4) = F \otimes q t^\varepsilon = u_2 t^\varepsilon, \\ \mathcal{S}(\Gamma_3) &= B \otimes q t^{2\varepsilon-\mu} = u_3 t^{2\varepsilon-\mu} \quad \text{with } \varepsilon = \frac{3\mu^2 + 3\mu + 1}{3(1 + 2\mu)}. \end{aligned}$$

Note that  $\mathcal{S}(\Lambda_{e_1}^1) = \mathcal{S}(\Gamma_4)^{-1}$ .

*Proof of Theorem 1.2 for odd Hirzebruch surfaces.* First notice that the polynomial  $M(u) = u^4 t^{2\mu} + u^3 t^\mu - t^{-1} \in \Pi^{\text{univ}}[u]$  has invertible main coefficient, so that for any positive integer  $\ell$ , there exist uniquely determined polynomials  $Q_\ell$  and  $R_\ell$  such that  $u^\ell t^{\ell\varepsilon} - 1 = M(u)Q_\ell(u) + R_\ell(u)$  and degree of  $R_\ell$  is smaller than degree of  $M$ . If Seidel’s morphism is not injective, then there exists  $\ell \in \mathbb{N} \setminus \{0\}$  such that  $R_\ell = 0$ .

Now, in view of the polynomials  $u^\ell t^{\ell\varepsilon} - 1$  and  $M$ , it is clear that proceeding to the long division of the former by the latter – which consists in  $\ell - 3$  steps – will produce a polynomial  $Q_\ell$  whose coefficients are finite linear combinations of powers of  $t$  (with rational coefficients). The fact that these linear combinations are finite ensures that  $Q_\ell$  induces a polynomial in  $\mathbb{Q}[u]$  when  $t$  is set to 1,  $Q_\ell^1$ , which satisfies  $u^\ell - 1 = (u^4 + u^3 - 1)Q_\ell^1(u)$  in  $\mathbb{Q}[u]$ .

Since the roots of  $u^4 + u^3 - 1$  are not roots of unity, we get a contradiction so that there are no non-zero integers  $\ell$  such that  $\mathcal{S}(x)^\ell = 1$ .  $\square$

3. 2-POINT BLOW-UPS OF  $S^2 \times S^2$ 

We now consider the manifold obtained from  $(M_\mu, \omega_\mu)$  by performing two successive symplectic blow-ups of capacities  $c_1$  and  $c_2$  with  $0 < c_2 \leq c_1 < c_1 + c_2 \leq 1 \leq \mu$ , which we denote by  $(M_{\mu, c_1, c_2}, \omega_{\mu, c_1, c_2})$ . In [5, Section 2.1], it is explained why this manifold is symplectomorphic to  $\mathbb{X}_3 = \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$  the 3-point blow-up of  $\mathbb{CP}^2$  with a symplectic form  $\omega_\nu$ , where  $\nu$  determines the cohomology class  $[\omega_\nu]$ , and why it is sufficient to consider values of  $c_1$  and  $c_2$  in the range above.

Let  $B, F \in H_2(M_{\mu, c_1, c_2}; \mathbb{Z})$  be the homology classes defined by  $B = [S^2 \times \{p\}]$ ,  $F = [\{p\} \times S^2]$  and let  $E_i \in H_2(M_{\mu, c_1, c_2}; \mathbb{Z})$  be the exceptional class corresponding to the blow-up of capacity  $c_i$ . We set  $u = (F - E_2) \otimes q$  and  $v = (B - E_2) \otimes q$ .

The following result was proved by Entov–Polterovich in [9].

**Lemma 3.1.** *As a  $\Pi^{\text{univ}}$ -algebra we have*

$$QH_*(M_{\mu, c_1, c_2}, \omega_{\mu, c_1, c_2}) \cong \Pi^{\text{univ}}[u, v]/I_\mu$$

where  $I_\mu$  is the ideal generated by

$$\begin{aligned} u^2 v^2 + u^2 v t^{-c_2} &= v t^{-\mu - c_2} + t^{c_1 - \mu - 1 - c_2} \text{ and} \\ u^2 v^2 + u v^2 t^{-c_2} &= u t^{-1 - c_2} + t^{c_1 - \mu - 1 - c_2}. \end{aligned}$$

*Proof.* We briefly recall the computation of this quantum algebra using the formalism of [4]. We follow the steps of the computation in Section 5.2 of that paper and only recall here what will be needed below to understand the non-injectivity result stated as Theorem 1.1. Consider the manifold  $M_{\mu, c_1, c_2}$  endowed with the standard action of the torus  $T = S^1 \times S^1$  for which the moment polytope is given by

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_2 \leq \mu, -1 \leq x_1 \leq 0, c_1 \leq x_2 - x_1 \leq \mu + 1 - c_2\}$$

so the primitive outward normals to  $P$  are as follows:

$$\eta_1 = (0, 1), \eta_2 = (1, 0), \eta_3 = (1, -1), \eta_4 = (0, -1), \eta_5 = (-1, 0), \text{ and } \eta_6 = (-1, 1).$$

The normalised moment map  $\Phi : M_{\mu, c_1, c_2} \rightarrow \mathbb{R}^2$  is given by

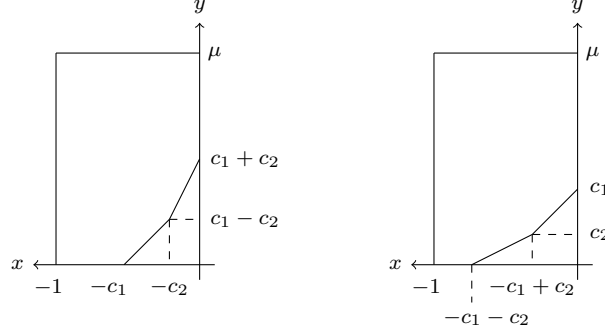
$$\Phi(z_1, \dots, z_6) = \left( -\frac{1}{2}|z_2|^2 + \epsilon_1, -\frac{1}{2}|z_1|^2 + \mu - \epsilon_2 \right)$$

where

$$(3) \quad \epsilon_1 = \frac{c_1^3 + 3c_2^2 - c_2^3 - 3\mu}{3(c_1^2 + c_2^2 - 2\mu)} \quad \text{and} \quad \epsilon_2 = \frac{c_1^3 - c_2^3 + 3c_2^2\mu - 3\mu^2}{3(c_1^2 + c_2^2 - 2\mu)}.$$

Moreover, the homology classes  $A_i = [\Phi^{-1}(D_i)]$  of the pre-images of the corresponding facets  $D_i$  are:  $A_1 = F - E_2$ ,  $A_2 = B - E_1$ ,  $A_3 = E_1$ ,  $A_4 = F - E_1$ ,  $A_5 = B - E_2$ , and  $A_6 = E_2$ . Let  $\Gamma_i$  be the circle action associated to  $\eta_i$ . Since this toric complex structure on  $M_{\mu, c_1, c_2}$  is Fano and  $T$ -invariant, it follows from [23, Theorem 1.10] or [4, Theorem 4.5] that the Seidel elements associated to these actions are given by the following expressions

$$\begin{aligned} \mathcal{S}(\Gamma_1) &= (F - E_2) \otimes q t^{\mu - \epsilon_2}, & \mathcal{S}(\Gamma_2) &= (B - E_1) \otimes q t^{\epsilon_1}, \\ \mathcal{S}(\Gamma_3) &= E_1 \otimes q t^{\epsilon_1 + \epsilon_2 - c_1}, & \mathcal{S}(\Gamma_4) &= (F - E_1) \otimes q t^{\epsilon_2}, \\ \mathcal{S}(\Gamma_5) &= (B - E_2) \otimes q t^{1 - \epsilon_1}, & \mathcal{S}(\Gamma_6) &= E_2 \otimes q t^{\mu + 1 - c_2 - \epsilon_1 - \epsilon_2}. \end{aligned}$$

FIGURE 1.  $(M_{\mu, c_1, c_2}, \omega_{\mu, c_1, c_2})$  with toric actions  $T_1$  and  $T_2$ .

There are nine primitive sets:  $\{1, 3\}$ ,  $\{1, 4\}$ ,  $\{1, 5\}$ ,  $\{2, 4\}$ ,  $\{2, 5\}$ ,  $\{2, 6\}$ ,  $\{3, 5\}$ ,  $\{3, 6\}$ ,  $\{4, 6\}$  which yield nine multiplicative relations (which form the Stanley–Riesner ideal) that, combined with the two linear relations ( $A_5 = A_1 + A_2 - A_4$  and  $A_6 = A_3 + A_4 - A_1$ ), give by [23, Proposition 5.2] the desired result.  $\square$

Assume from now on that  $\mu = 1$ . Recall from [5] that if  $c_2 < c_1$  then

$$\pi_1(\text{Ham}(M_{\mu, c_1, c_2}, \omega_{1, c_1, c_2})) = \mathbb{Z}\langle x_0, x_1, y_0, y_1, z \rangle \simeq \mathbb{Z}^5$$

where the generators  $x_0, x_1, y_0, y_1, z$  correspond to circle actions contained in maximal tori of the Hamiltonian group. For example, we have  $x_0 = \Gamma_2$  and  $y_0 = \Gamma_1$  where the  $\Gamma_i$ 's are defined in the proof of the lemma above. In order to understand the remaining generators consider the two toric manifolds given by the polytopes in Figure 1 and let us denote by  $\{x_{0,i}, y_{0,i}\}$  the generators in  $\pi_1(T_i)$ , where  $T_i$ ,  $i = 1, 2$ , represent the two torus actions in this figure and the generators  $\{x_{0,i}, y_{0,i}\}$  correspond to the circle actions whose moment maps are, respectively, the first and second components of the moment map associated to each one of the toric actions. It was shown in [5, Section 4.2] that  $x_1 = x_{0,1}$ ,  $z = y_{0,2}$  and  $y_1 = y_{0,1} - x_1 = z - x_{0,2}$ . The case  $c_1 = c_2$  is an interesting limit case in terms of the topology of the Hamiltonian group since  $y_1$  disappears. For more details see [5].

To prove Theorem 1.1, we will show that

**Proposition 3.2.** *The class of  $2(x_0 + y_0)$  belongs to  $\ker(\mathcal{S})$  if and only if  $c_1 = c_2$ .*

*Proof.* From the computation of the Seidel elements in the proof of the lemma above one gets that in the general case (i.e  $\mu \geq 1$ )  $\mathcal{S}(x_0) = v^{-1}t^{\varepsilon_1-1}$  and  $\mathcal{S}(y_0) = ut^{\mu-\varepsilon_2}$ . As the Seidel elements are invertible, this yields invertibility of  $u$  and  $v$ .

Since  $\mu \geq 1 > c_2^2$ , it is straightforward to deduce from (3) that  $\varepsilon_1 = \varepsilon_2$  if and only if  $\mu = 1$ : we now restrict our attention to this case and denote by  $\varepsilon$  the common value of  $\varepsilon_1 = \varepsilon_2$ . By invertibility of  $u$  and  $v$ , the fact that  $2(x_0 + y_0)$  belongs to  $\ker(\mathcal{S})$  is equivalent to  $u^2 = v^2$ . Note that the relations in  $I_1$  are equivalent to

$$u^2 = t^{-1} + v^{-1}t^{c_1-2} - u^2vt^{c_2} \quad \text{and} \quad v^2 = t^{-1} + u^{-1}t^{c_1-2} - uv^2t^{c_2}$$

so that  $u^2 = v^2$  is equivalent to

$$v^{-1}t^{c_1-2} - u^2vt^{c_2} = u^{-1}t^{c_1-2} - uv^2t^{c_2}.$$



Using one more time the relations in  $I_1$  to replace the terms  $u^2vt^{c_2}$  and  $uv^2t^{c_2}$  in the previous equality, we obtain

$$v^{-1}t^{c_1-1} + ut^{c_2} = u^{-1}t^{c_1-1} + vt^{c_2}.$$

The relations in  $I_1$  also induce, by subtracting one from the other, the equality  $(v^{-1} - u^{-1})t^{-1} = v - u$  so we conclude that  $u^2 = v^2$  if and only if  $(u - v)(t^{c_1} - t^{c_2}) = 0$  which is equivalent to  $c_1 = c_2$  as otherwise  $t^{c_1} - t^{c_2}$  is invertible.  $\square$

## REFERENCES

- [1] M. Abreu and D. McDuff, *Topology of symplectomorphism groups of rational ruled surfaces*, J. Amer. Math. Soc. **13** (2000), 971–1009 (electronic).
- [2] M. Abreu, G. Granja and N. Kitchloo, *Compatible complex structures on symplectic rational ruled surfaces*, Duke Math. Journal, **148** (2009), 539–600.
- [3] P. Albers, *A Lagrangian Piunikhin–Salamon–Schwarz morphism and two comparison homomorphisms in Floer homology*, Int. Math. Res. Notices (2008), doi:10.1093/imrn/rnm134.
- [4] S. Anjos and R. Leclercq, *Seidel's morphism of toric 4-manifolds*, J. Symplectic Geom. (accepted Dec 2015), arXiv:1406.7641 (2014).
- [5] S. Anjos and M. Pinsonnault, *The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of the projective plane*, Math. Z. **275** (2013), 245–292.
- [6] J.-F. Barraud and O. Cornea, *Lagrangian intersections and the Serre spectral sequence*, Ann. of Math. (2) **166** (2007), 657–722.
- [7] J.-F. Barraud and O. Cornea, *Higher order Seidel invariants for loops of hamiltonian isotopies*, in preparation.
- [8] K. Chan, S.-C. Lau, N C Leung, and H.-H. Tseng, *Open Gromov–Witten invariants, mirror maps, and Seidel's representations for toric manifolds*, arXiv:1209.6119 (2012).
- [9] M. Entov and L. Polterovich, *Symplectic quasi-states and semi-simplicity of quantum homology*. In *Toric topology*, vol. 460 of Contemp. Math., Amer. Math. Soc., Providence, RI (2008), 47–70.
- [10] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds*, arXiv:1009.1648 (2010).
- [11] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Spectral invariants with bulks, quasi-morphisms, and Lagrangian Floer theory*, arXiv:1105.5123 (2011).
- [12] E. González and H. Iritani, *Seidel elements and mirror transformations*, Selecta Math. (N.S.) **18** (2012), no. 3, 557–590.
- [13] S. Hu and F. Lalonde, *A relative Seidel morphism and the Albers map*, Trans. amer. Math. Soc. **362** (2010), 1135–1168.
- [14] S. Hu, F. Lalonde, and R. Leclercq, *Homological Lagrangian monodromy*, Geom. Topol. **15** (2011), 1617–1650.
- [15] M. Hutchings, *Floer homology for families. I*, Algebr. Geom. Topol. **8** (2008), 435–492.
- [16] Y. Karshon, *Periodic Hamiltonian Flows on Four Dimensional Manifolds*, Mem. Amer. Math. Soc. **141**, no. 672 (1999).
- [17] F. Lalonde, D. McDuff, and L. Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*, Invent. Math. **135** (1999), 369–385.
- [18] R. Leclercq, *Spectral invariants in Lagrangian Floer theory*, J. Mod. Dyn. **2** (2008), 249–286.
- [19] R. Leclercq, *The Seidel morphism of cartesian products*, Algebr. Geom. Topol. **9** (2009), 1951–1969.
- [20] R. Leclercq, and F. Zapolsky, *Spectral invariants for monotone Lagrangians*, arXiv:math.SG/1505.07430 (2015).
- [21] D. McDuff, *Loops in the Hamiltonian group: a survey*, in *Symplectic topology and measure preserving dynamical systems*, Contemp. Math. **512**, 127–148, Amer. Math. Soc., Providence, RI, (2010).
- [22] D. McDuff and D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Amer. Mat. Soc., Providence, RI (2004).

- [23] D. McDuff and S. Tolman, *Topological properties of Hamiltonian circle actions*, Int. Math. Res. Papers (2006), doi:10.1155/IMRP/2006/72826.
- [24] Y. Savelyev, *Quantum characteristic classes and the Hofer metric*, Geom. Topol. **12** (2008), 2277–2326.
- [25] P. Seidel,  $\pi_1$  of symplectic automorphism groups and invertibles in quantum homology rings, Geom. Funct. Anal. **7** (1997), 1046–1095.

SA: CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS, MATHEMATICS DEPARTMENT, INSTITUTO SUPERIOR TÉCNICO, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL

*E-mail address:* `sanjos@math.ist.utl.pt`

RL: LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UNIV. PARIS-SUD, CNRS, UNIVERSITÉ PARIS-SACLAY, 91405 ORSAY, FRANCE.

*E-mail address:* `remi.leclercq@math.u-psud.fr`